Aequat. Math. 90 (2016), 607–623

0001-9054/16/030607-17
published online November 26, 2015
DOI 10.1007/s00010-015-0387-6

Aequationes Mathematicae

Coplanarity of lines in projective and polar Grassmann spaces

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Abstract. The structure of lines in a projective or a polar Grassmann space together with binary coplanarity is a sufficient system of primitive notions for these geometries.

Mathematics Subject Classification. 51A45, 51A50.

Keywords. Grassmann space, projective space, polar space, coplanarity, bundle space.

Introduction

In geometry a line is usually identified with the set of all points that are incident with that line. Dually, a point can be identified with the set of all lines incident with that point, i.e. with a bundle of lines. As a matter of fact, the geometry of an incidence structure remains unaffected if we interchange points with lines. In other words the point-line and the line-bundle structure, called a *bundle space*, are equivalent. Points together with ternary collinearity is a sufficient system of primitive notions for a partial linear space, as well as, lines with ternary concurrency.

A natural question arises if there is a sufficient binary relation. As we enter into geometries on the universe of lines we can see a lot of contributions to this topic. It was Pieri who first proved that projective three-dimensional space can be axiomatized in terms of lines and binary line intersection [27]. His axiom system has been improved in [14] and independently two other systems have been presented in [16, 33]. In [30, Ch. 7] we can find yet another axiom system. Axiomatization for higher dimensions is given in [23]. Affine spaces, except those where all lines are of size 2, can be axiomatized by means of line intersection as a sole primitive notion [13]. In view of [29] binary line intersection is sufficient to express the geometry of spine spaces.

More in vein of mild hypotheses characterizations of geometrical mappings two papers [9, 10] give an account on bijective transformations of lines preserving line intersection in projective and affine spaces. In metric geometry lines intersecting at right angles play an essential role. It has been proved in [8,11] for elliptic spaces, in [12] for symplectic spaces, and in [19–22,24] for hyperbolic spaces that transformations which preserve ortho-adjacency of lines are induced by collineations that preserve orthogonality, unless the underlying projective space has three dimensions. These results were generalized for k-subspaces in metric-projective settings in [28]. Transformations of lines that preserve ortho-adjacency in Euclidean spaces were investigated in [2–5] as well as in [17,31].

In most geometries line intersection is equivalent to coplanarity of lines, but it is not in Grassmann spaces. In this paper we reconstruct the geometry of projective and polar Grassmann spaces from their line structures equipped with a binary coplanarity relation (cf. Theorem 1.1). We try to imitate the approach of [18] and other papers, e.g. [1,15,34], where the construction of a bundle space is used to recover various geometries embeddable into a projective space. However, we do not use it to settle the problem of projective embeddability.

The idea of reconstruction is simple. Let $\mathfrak{M}$ be a projective or a polar Grassmann space. It is locally projective in that its strong subspaces are projective spaces. A point of $\mathfrak{M}$ is identified with the bundle of lines passing through this point. We prove that there are only two types of maximal coplanarity cliques in $\mathfrak{M}$. Those not planar are called *semibundles* and the problem is that they are not complete bundles of lines. A semibundle is the portion of a bundle contained in a maximal strong subspace of $\mathfrak{M}$. Therefore, one of the essential tasks was to group semibundles into bundles in terms of coplanarity relations. Roughly speaking, this is the way we prove our main theorem.

The notion of a pencil of lines, the set of lines through a point on a plane, emerges when we want to geometrically distinguish two types of maximal coplanarity cliques in $\mathfrak{M}$. Such a clique with the structure of pencils of lines is a projective space. This lets us distinguish two types of cliques in $\mathfrak{M}$ by their geometrical dimension. We can do this, though, only under suitable assumptions on the dimension of maximal strong subspaces in $\mathfrak{M}$ (cf. (12)). So, there appear cases when our bundle technique fails. Nevertheless, in all but one of them we are still able to recover $\mathfrak{M}$ by Chow's Theorem [6].

In the last section we discuss connections with the Bundle Theorem (cf. Theorem 4.5) introduced in [34]. It holds true in projective Grassmann spaces whereas unexpectedly it is generally false in polar Grassmann spaces (cf. Remark 4.6).

Similarly, the question can be asked whether it is possible to axiomatize, by means of coplanarity of lines, geometries which are locally affine or semiaffine, e.g. spine spaces where strong subspaces are projective, affine or semiaffine. This is however the topic for another paper.

1. General Grassmann spaces

Let us start with a rather general definition of a Grassmann space. We follow the ideas of [26]. Let $\langle P, \subseteq \rangle$ be a poset and let $\dim\colon P \longrightarrow \{0, 1, \dots, n\}$ be a dimension function. For $0 \leq k \leq n$, we denote by P_k the family of all k-dimensional members of P. For $Z, Y \in P$ a *k-interval* is a set

$$[Z, Y]_k := \big\{U \in P_k \colon Z \subseteq U \subseteq Y\big\}. \tag{1}$$

Its variation for $H \in P_{k-1}$ and $B \in P_{k+1}$ with $H \subseteq B$ is

$$\mathbf{p}(H, B) := [H, B]_k \tag{2}$$

which we call a *k-pencil*. It can be viewed as a generalization of the well known *pencil of lines*, that is, a set of lines through a fixed point H on a plane B. The point-line structure

$$\mathbf{P}_k(P) := \big\langle P_k, \mathcal{P}_k(P)\big\rangle, \tag{3}$$

where $\mathcal{P}_k(P)$ is the family of all k-pencils, is a *Grassmann space* (cf. [25, 32]). At this level of generality not much can be said about the properties of this structure. In the classical approach P is the family of subspaces of some projective space, or equivalently, of a vector space when $n \geq 3$. In this paper we also deal with the family of subspaces of a polar space as P.

Now, let S be an arbitrary set and $\mathcal{L} \subseteq 2^S$. We call the elements of S *points* and the elements of $\mathcal{L}$ *lines*. Then the point-line structure $\mathfrak{M} = \langle S, \mathcal{L} \rangle$ is a *partial linear space* whenever two distinct lines meet in at most one point. The set of all lines through a given point is a *bundle of lines*. A subset $X \subseteq S$ is called a *subspace* of $\mathfrak{M}$ if every line that joins two distinct points in X is entirely contained in X. A subspace is said to be *strong* if every two of its points are collinear. A *plane* in $\mathfrak{M}$ is a strong subspace E of $\mathfrak{M}$ with the property that the restriction of $\mathfrak{M}$ to E is a projective plane. For lines $L_1, L_2 \in \mathcal{L}$ we say that they are *coplanar* and write

$$L_1 \,\boldsymbol{\pi}\, L_2 \quad \text{iff} \quad \text{there is a plane } E \text{ such that } L_1, L_2 \subset E. \tag{4}$$

In this paper we deal with the structure $\langle \mathcal{L}, \boldsymbol{\pi} \rangle$ and our goal is to prove the following theorem.

Theorem 1.1. *Let $\mathfrak{M}$ be a projective or a polar Grassmann space and let $\mathcal{L}$ be its line set. If maximal strong subspaces of $\mathfrak{M}$ are at least planes and neither all of them are planes nor all of them are projective 3-spaces, then $\mathfrak{M}$ and the structure of its lines together with the coplanarity relation $\langle \mathcal{L}, \boldsymbol{\pi} \rangle$ are definitionally equivalent.*

In other words, we can characterize the automorphisms of projective and polar Grassmann spaces as coplanarity-preserving bijections. Namely the following holds.

Corollary 1.2. *Under the assumptions of 1.1, every bijective transformation of the line set $\mathcal{L}$ preserving the coplanarity relation $\boldsymbol{\pi}$ in both directions is induced by a collineation of $\mathfrak{M}$.*

Before we enter into details of the two aforementioned cases we are going to prove some basic facts in these general settings. It is not our intention however, to provide an axiom system embracing both projective and polar Grassmann spaces (for an axiomatic approach to Grassmann spaces see [32]). The following assumptions are just what we need in our reasonings.

We assume that $\mathfrak{M}$ is a Gamma space, i.e. it satisfies the so called *none-one-or-all axiom* stating that a point not on a line is collinear with none, one or all the points on that line (cf. [7]). We also assume that there are two disjoint nonempty classes $\mathcal{S}$, $\mathcal{T}$ of maximal strong subspaces in $\mathfrak{M}$ that are also maximal cliques of binary collinearity of $\mathfrak{M}$. The elements of $\mathcal{S}$ will be called *stars* and the elements of $\mathcal{T}$ will be called *tops*. These two families have to meet the following requirements:

(A1) For $X \in \mathcal{S} \cup \mathcal{T}$ the restriction of $\mathfrak{M}$ to X is a projective space. As such all stars have the same dimension and all tops have the same dimension.
(A2) Maximal strong subspaces in $\mathfrak{M}$ are at least planes. If stars are planes, then tops are at least projective 3-spaces and vice versa.
(A3) The intersection of a star and a top is empty, a point, or a line.
(A4) If X_1, X_2 are a star and a top such that $X_1 \cap X_2 \neq \emptyset$ and $U_1 \in X_1$ is collinear with $U_2 \in X_2$, then $U_1 \in X_1 \cap X_2$ or $U_2 \in X_1 \cap X_2$.

1.1. Coplanarity cliques

For a subspace X of $\mathfrak{M}$ we write

$$\mathrm{L}(X) = \{L \in \mathcal{L} \colon L \subset X\}. \tag{5}$$

If U is a point in a subspace X, then we write

$$\mathrm{L}_U(X) = \{L \in \mathcal{L} \colon U \in L \subseteq X\}. \tag{6}$$

If E is a plane in $\mathfrak{M}$, then $L(E)$ is a $\boldsymbol{\pi}$-clique. We call such cliques *flats*. If X is a strong subspace of $\mathfrak{M}$ and U is a point in X, then $\mathrm{L}_U(X)$ is a different kind of $\boldsymbol{\pi}$-clique which we call a *semibundle*. Semibundles play an essential role in our proof of 1.1. Actually we need semibundles in stars or tops. Note that if X is a plane then $\mathrm{L}_U(X) \subset \mathrm{L}(X)$. Such a semibundle is never a maximal $\boldsymbol{\pi}$-clique. For this reason we assume in (A2) that stars or tops are at least projective 3-spaces.

Lemma 1.3. *Every maximal $\boldsymbol{\pi}$-clique is either a flat or a semibundle.*

Proof. Let K be a maximal $\boldsymbol{\pi}$-clique. We have $|K| \geq 3$ as planes in $\mathfrak{M}$ are projective planes. So, take three pairwise distinct lines $L_1, L_2, L_3 \in K$. The lines L_1, L_2 span a plane E and meet in some point U.

In case $U \notin L_3$ the line L_3 lies in E. The lines L_1, L_2, L_3 form a triangle and thus all the other lines from K lie in E. Therefore, K is a flat.

In case $U \in L_3$ either $L_3 \subseteq E$ or $L_3 \nsubseteq E$. If $L_3 \subseteq E$, then we extend E to a maximal strong subspace X. If $X = E$, then K is a flat. Otherwise we have three pairwise distinct lines through U not all on the plane E, likewise in the case where $L_3 \nsubseteq E$. So, without loss of generality we can assume now that $L_3 \nsubseteq E$. We will show that there is a maximal strong subspace X containing L_1, L_2, L_3. Take $U_3 \in L_3$ with $U_3 \neq U$. There are points $U_1 \in L_1$, $U_2 \in L_2$ both distinct from U and collinear with U_3. The points U, U_1, U_2 form a triangle on E. As $\mathfrak{M}$ is a Gamma space U_3 is collinear with all the points on $\overline{U_1, U_2}$. So, it is collinear with all the points on all sides of the triangle U, U_1, U_2 since $L_3 \,\boldsymbol{\pi}\, L_1, L_2$. In consequence, U_3 must be collinear with all the points on E. Therefore, L_3 lies in some maximal clique of binary collinearity containing E, say X, which is a maximal strong subspace of $\mathfrak{M}$. By (A1) all the lines through U in X are in K and every line in K goes through U. Hence K is a semibundle. $\square$

We write $\mathcal{K}$ for the family of all maximal $\boldsymbol{\pi}$-cliques. Since every plane lies in some maximal strong subspace, considering that we have two types of maximal strong subspaces, $\mathcal{K}$ is the union of four classes of $\boldsymbol{\pi}$-cliques: star flats, top flats, star semibundles and top semibundles.

Besides the characterization of maximal $\boldsymbol{\pi}$-cliques provided in Lemma 1.3 we need an elementary definition of such cliques within the structure $\langle \mathcal{L}, \boldsymbol{\pi} \rangle$. For lines $L_1, L_2, L_3 \in \mathcal{L}$ we define

$$\begin{aligned} \boldsymbol{\Delta_\pi}(L_1, L_2, L_3) \quad \text{iff} \quad & \neq (L_1, L_2, L_3) \text{ and } \boldsymbol{\pi}\,(L_1, L_2, L_3) \quad \text{and} \\ & \text{for all } M_1, M_2 \in \mathcal{L} \text{ if } M_1, M_2 \,\boldsymbol{\pi}\, L_1, L_2, L_3 \quad \text{then} \quad M_1 \,\boldsymbol{\pi}\, M_2, \end{aligned} \tag{7}$$

which says that the lines L_1, L_2, L_3 span a $\boldsymbol{\pi}$-clique. The $\boldsymbol{\pi}$-clique spanned by the lines L_1, L_2, L_3 is then the set

$$[L_1, L_2, L_3]_{\boldsymbol{\pi}} := \big\{ L \in \mathcal{L} \colon L \,\boldsymbol{\pi}\, L_1, L_2, L_3 \big\}. \tag{8}$$

There has been proved a bit more than stated in Lemma 1.3, namely

$$\mathcal{K} = \big\{ [L_1, L_2, L_3]_{\boldsymbol{\pi}} \colon L_1, L_2, L_3 \in \mathcal{L} \text{ and } \boldsymbol{\Delta_\pi}(L_1, L_2, L_3) \big\}. \tag{9}$$

Lemma 1.4. *The family $\mathcal{K}$ of maximal $\boldsymbol{\pi}$-cliques is definable in $\langle \mathcal{L}, \boldsymbol{\pi} \rangle$.*

1.2. Grouping semibundles into bundles

The idea of the proof of Theorem 1.1 is simple: to reconstruct the point set of $\mathfrak{M}$ in $\langle \mathcal{L}, \boldsymbol{\pi} \rangle$ by identifying bundles of lines with their vertices. The problem is that we have semibundles so far. We need to group all semibundles that correspond to the same point in $\mathfrak{M}$. This is where the relation Υ comes into play. Let $K_i := \mathrm{L}_{U_i}(X_i)$ be a semibundle for some strong subspace $X_i \in \mathcal{S} \cup \mathcal{T}$ and a point $U_i \in X_i$, where $i = 1, 2$. We write

$$\Upsilon(K_1, K_2) \quad \text{iff} \quad \text{for every } L_1 \in K_1 \text{ there is } L_2 \in K_2 \text{ such that } L_1 \boldsymbol{\pi} L_2. \tag{10}$$

The following is somewhat technical but it is a key fact when it comes to Υ.

Lemma 1.5. (i) *If $\Upsilon(K_1, K_2)$, then every line in K_1 meets X_2.*
(ii) *If $\Upsilon(K_1, K_2)$, then $X_1 \cap X_2 \neq \emptyset$.*
(iii) *If $\Upsilon(K_1, K_2)$, then $U_2 \in X_1$.*
(iv) *If $\Upsilon(K_1, K_2)$, then U_1, U_2 are collinear in $\mathfrak{M}$.*
(v) *If $\Upsilon(K_1, K_2)$ and $U_1 \notin X_2$, then $X_1 \cap X_2$ is a hyperplane in X_1.*

Proof. (i) Let $L_1 \in K_1$. There is $L_2 \in K_2$ such that L_1, L_2 are coplanar in $\mathfrak{M}$. Hence L_1 meets L_2 in some point of X_2.
(ii) The point of meeting in (i) is actually in $X_1 \cap X_2$.
(iii) Let $U \in X_1 \backslash X_2$. The line $L_1 = \overline{U_1, U}$ is coplanar with some line $L_2 \in K_2$. Hence the points U, U_1, U_2 lie on a plane, which implies that U, U_2 are collinear. In consequence all the points of X_1 are collinear with U_2. This means that $U_2 \in X_1$ as X_1 is a maximal clique of binary collinearity in $\mathfrak{M}$.
(iv) Immediate consequence of (iii).
(v) Condition (A2) provides the existence of a line L in X_1 not through U_1. Take two distinct points D_1, D_2 on L. In view of (A1) we can think of X_1 as a projective space so, by (i) the lines $\overline{U_1, D_1}$, $\overline{U_2, D_2}$ meet X_2 in points W_1, W_2 respectively, which are distinct. Note that the lines L and $\overline{W_1, W_2}$ are coplanar and thus they meet each other. As $\overline{W_1, W_2} \subseteq X_2$ the line L meets X_2. Considering (i) every line of X_1 meets $X_1 \cap X_2$, which means that $X_1 \cap X_2$ is a hyperplane in X_1.

□

Lemma 1.6. *If X_1, X_2 are a star and a top, then*

$$\Upsilon(K_1, K_2) \quad \textit{and} \quad \Upsilon(K_2, K_1) \quad \textit{iff} \quad X_1 \cap X_2 \textit{ is a line and } U_1, U_2 \in X_1 \cap X_2.$$

Proof. $\Rightarrow$: In view of (A3) a star and a top can share at most a line. So, $X_1 \cap X_2$ is a point or a line by Lemma 1.5(ii). By Lemma 1.5(iii) we get $U_1, U_2 \in X_1 \cap X_2$.

Suppose that $X_1 \cap X_2$ is a point. Then $U := U_1 = U_2$. Consider a point $W_1 \in X_1 \backslash X_2$. The line $\overline{U, W_1}$ is coplanar with a line $\overline{U, W_2}$ for some $W_2 \in X_2 \backslash X_1$. But this means that W_1, W_2 are collinear which is impossible by (A4).

$\Leftarrow$: Since X_1, X_2 are projective spaces by (A1), every line through U_1 in X_1, i.e. every line in K_1, and every line through U_2 in X_2, i.e. every line in

K_2, is coplanar with the line $X_1 \cap X_2 \in K_1 \cap K_2$. Consequently $\Upsilon(K_1, K_2)$ and $\Upsilon(K_2, K_1)$. $\square$

1.3. Flats versus semibundles

Let $K_1 := \mathrm{L}(E)$ for some plane E that lies in a maximal strong subspace E' and $K_2 := \mathrm{L}_U(X)$ for some maximal strong subspace X through a point U. The intersection of the flat K_1 with the semibundle K_2 can be empty, a single line, or a pencil of lines. Indeed, by (A3) the second case arises when $U \in E$, E' is of different type than X, and $E \cap X$ is a line. We get a pencil of lines when $U \in E \subseteq X$, that is when both E' and X are of the same type. The intersection of two distinct flats is empty or a line. The situation is completely different in the case of semibundles. The intersection of two distinct semibundles can be empty or a line if their vertices are distinct, otherwise the intersection is again a semibundle, in particular it can be a single line or a pencil of lines.

Every pencil of lines of $\mathfrak{M}$ determines a flat uniquely. The assumptions of (A2) let us extend such a pencil of lines to a semibundle which is a maximal $\boldsymbol{\pi}$-clique only when the dimensions permit. Therefore, the natural definition of a pencil of lines in $\langle \mathcal{L}, \boldsymbol{\pi} \rangle$ as the intersection of two maximal $\boldsymbol{\pi}$-cliques of two types (frankly *flat semibundle*) has some flaw and does not cover all the pencils of lines in $\mathfrak{M}$. That is to say not all pencils of lines in $\mathfrak{M}$ are flat semibundles in $\langle \mathcal{L}, \boldsymbol{\pi} \rangle$. Anyway, let $\mathcal{P}$ be the family of all minimal elements of the set

$$\big\{ K_1 \cap K_2 \colon K_1, K_2 \in \mathcal{K},\ K_1 \neq K_2,\ |K_1 \cap K_2| \geq 2 \big\}. \tag{11}$$

The above analysis lets us state the following.

Lemma 1.7. *The family $\mathcal{P}$ defined in $\langle \mathcal{L}, \boldsymbol{\pi} \rangle$ coincides with the family of those pencils of lines in $\mathfrak{M}$ which can be extended to semibundles that are maximal $\boldsymbol{\pi}$-cliques.*

Note that a maximal $\boldsymbol{\pi}$-clique together with the pencils of lines it contains carries the structure of a projective space. The geometrical dimension of a flat is always 2 whereas a semibundle $\mathrm{L}_U(X)$ has dimension one less than the dimension of X. This lets us distinguish flats from semibundles if we assume that

$$\text{stars or tops in } \mathfrak{M} \text{ are at least projective 4-spaces.} \tag{12}$$

In other words, (12) excludes the following cases: tops and stars are planes, tops are planes and stars are 3-spaces (and vice versa), tops and stars are 3-spaces. In all these cases we cannot distinguish flats from semibundles, but only in the first case is none of the pencils of lines definable in $\langle \mathcal{L}, \boldsymbol{\pi} \rangle$. If say tops are 3-spaces and stars are 4-spaces, then top semibundles are projective planes while star semibundles are 3-spaces. So, in terms of $\boldsymbol{\pi}$ we are still not able to distinguish top semibundles from flats, but this is no problem as star semibundles are sufficient in this case.

By Lemma 1.7 for a $\boldsymbol{\pi}$-clique K from the set

$$\mathcal{K}_{\mathcal{P}} := \big\{K \in \mathcal{K}\colon \text{there is } q \in \mathcal{P} \text{ such that } q \subset K\big\} \tag{13}$$

we can define its geometrical dimension $\dim(K)$. This lets us make the following definition

$$\mathcal{B} := \big\{K \in \mathcal{K}_{\mathcal{P}}\colon \dim(K) \geq 3\big\}. \tag{14}$$

In view of (12) we obtain:

Lemma 1.8. *The family $\mathcal{B}$ defined in $\langle \mathcal{L}, \boldsymbol{\pi}\rangle$ coincides with the family of all top semibundles, the family of all star semibundles or the union of these two families depending on whether tops, stars or all of them are at least projective 4-spaces.*

2. Projective Grassmann spaces

Let V be a (left) vector space of dimension $n < \infty$ over a division ring. We denote by $\mathrm{Sub}(V)$ the family of all subspaces, and by $\mathrm{Sub}_k(V)$ the family of all k-dimensional subspaces of V. In (3) take $P = \mathrm{Sub}(V)$. Then $\mathfrak{M} := \mathbf{P}_k(V)$ is a *projective Grassmann space*. It is a Gamma space (cf. [7]). If $k \leq 1$, or $k \geq n-1$, then $\mathfrak{M}$ is a projective space (with precisely one point for $k = 0, n$). In the other cases $\mathfrak{M}$ is a proper partial linear space, that is there are pairs of noncollinear points in it.

There are two disjoint classes of maximal strong subspaces in $\mathfrak{M}$: stars of the form $[H, V]_k$, where $H \in \mathrm{Sub}_{k-1}(V)$, and tops of the form $[\Theta, B]_k$, where $B \in \mathrm{Sub}_{k+1}(V)$ and Θ is the zero subspace of V. A plane in $\mathfrak{M}$ is a k-interval

- $[H, Y]_k$, where $H \subseteq Y$, $H \in \mathrm{Sub}_{k-1}(V)$ and $Y \in \mathrm{Sub}_{k+2}(V)$ or
- $[Z, B]_k$, where $Z \subseteq B$, $Z \in \mathrm{Sub}_{k-2}(V)$ and $B \in \mathrm{Sub}_{k+1}(V)$.

Note that stars as projective spaces have dimension $n-k$ whereas tops have dimension k. So, assume that

$$3 < k \text{ and } k < n-1 \quad \text{or} \quad 1 < k \ \text{ and } \ k < n-3 \tag{15}$$

to exclude the following cases: $(k, n) \in \{(2,4), (2,5), (3,5), (3,6)\}$. Then $\mathfrak{M}$ satisfies conditions (A1)–(A4) and we can apply Lemmas 1.3, 1.5, and 1.6 here. Moreover, (12) allows us to distinguish flats from semibundles. On a side note, a maximal $\boldsymbol{\pi}$-clique in $\mathfrak{M}$ uniquely determines a strong subspace containing this clique. Two distinct tops, as well as two distinct stars, can share at most a point. A star and a top are either disjoint or share a line.

Our goal is to group lines into bundles, so that bundles can be identified with points. In view of Lemma 1.6, the relation Υ groups too many cliques including those which have different vertices. In order to avoid this we additionally require that the cliques share no line. To preserve reflexivity we allow the cliques to be equal. The way that Lemma 1.6 is formulated indicates that the relation we need must be symmetric. Finally, for semibundles K_1, K_2 we write

$$\Upsilon_\emptyset(K_1, K_2) \quad \text{iff} \quad \Upsilon(K_1, K_2), \Upsilon(K_2, K_1), \text{ and either} \\ K_1 \cap K_2 = \emptyset \quad \text{or} \quad K_1 = K_2. \tag{16}$$

Lemma 2.1. *The following conditions are equivalent:*

(i) $\Upsilon_\emptyset(K_1, K_2)$,
(ii) X_1, X_2 *are of the same type and* $U_1 = U_2$.

Proof. (i) $\implies$ (ii): In view of Lemma 1.6 and the requirements of (16) X_1, X_2 are of the same type. If $X_1 = X_2$, then K_1, K_2 are two bundles in some projective space. Therefore, it is clear that $U_1 = U_2$.

Assume that $X_1 \neq X_2$. By Lemma 1.5(iii) we get $U_1, U_2 \in X_1 \cap X_2$. Consequently, $U_1 = U_2$ as two distinct maximal strong subspaces of the same type can share at most one point.

(ii) $\implies$ (i): In case $X_1 = X_2$ we have $K_1 = K_2$ and thus $\Upsilon(K_1, K_2)$.

Now, assume that $X_1 \neq X_2$ and set $U := U_1 = U_2$. Let $L_1 \in K_1$. Take the extension X of L_1 to a maximal strong subspace of the other type than X_1, X_2. We see that $U \in X_2 \cap X$. Hence $L_2 := X_2 \cap X$ must be a line. Observe that $L_2 \in K_2$. The lines L_1, L_2 meet each other in U and both lie in the projective space carried by X. Thus they span a plane in X. Consequently $\Upsilon(K_1, K_2)$. Nothing changes when we change indices, hence $\Upsilon(K_2, K_1)$.

To make the proof complete, note that maximal strong subspaces of the same type can share at most a single point, thus $K_1 \cap K_2 = \emptyset$ unless $K_1 = K_2$. $\square$

For $K \in \mathcal{B}$ we make the following definition

$$\Lambda_{\Upsilon_\emptyset}(K) := \bigcup \{ K' \in \mathcal{B} \colon \Upsilon_\emptyset(K, K') \}. \tag{17}$$

Lemma 2.2. *If U is a point and X is a maximal strong subspace with $U \in X$, then*

$$\Lambda_{\Upsilon_\emptyset}(\mathrm{L}_U(X)) = \{ L \in \mathcal{L} \colon U \in L \}. \tag{18}$$

Proof. It suffices to show that the set on the left hand side contains all the lines through U. So, let L be a line through U. Extend L to a maximal strong subspace Y of the same type as X. Then $L \in \mathrm{S}_Y(U)$. By Lemma 2.1 we get $\Upsilon_\emptyset(\mathrm{L}_U(X), \mathrm{S}_Y(U))$ which completes the proof. $\square$

We have actually proved that $\Lambda_{\Upsilon_\emptyset}(K)$ is a bundle for arbitrary $K \in \mathcal{B}$. It can be identified with its vertex. Note that bundles $K_1, K_2, \dots K_r$ are collinear iff $\Lambda_{\Upsilon_\emptyset}(K_1) \cap \Lambda_{\Upsilon_\emptyset}(K_2) \cap \dots \cap \Lambda_{\Upsilon_\emptyset}(K_r) \neq \emptyset$. This is enough to state our main Theorem 1.1 for projective Grassmann spaces.

3. Polar Grassmann spaces

Let ξ be a nondegenerate reflexive sesquilinear form of index m on V of dimension $n \leq \infty$. For $U, W \in \mathrm{Sub}(V)$ we write $U \perp W$ iff $\xi(U, W) = 0$, meaning that $\xi(u, w) = 0$ for all $u \in U$, $w \in W$. Then the set of all totally isotropic subspaces of V is

$$\mathrm{Q} := \{U \in \mathrm{Sub}(V) \colon U \perp U\},$$

and $\mathrm{Q}_k := \mathrm{Q} \cap \mathrm{Sub}_k(V)$. The set Q_k is nonempty iff $k \leq m$. The point-line structure $\langle \mathrm{Q}_1, \mathrm{Q}_2 \rangle$ is a polar space. Taking $P = \mathrm{Q}$ in (3) we get a *polar Grassmann space* $\mathfrak{M} := \mathbf{P}_k(\mathrm{Q})$. The key observation is that tops in polar Grassmann spaces are of the same form as in projective Grassmann spaces, i.e. $[\Theta, B]_k$ where $B \in \mathrm{Q}_{k+1}$, but stars are k-intervals $[H, Y]_k$, where $H \in \mathrm{Q}_{k-1}$, $Y \in \mathrm{Q}_m$, and $H \subseteq Y$. So, it is seen that stars as projective spaces are $m - k$ dimensional while tops are k dimensional. If we assume that

$$3 < k \ \text{ and } \ k < m - 1 \qquad \text{or} \qquad 1 < k \ \text{ and } \ k < m - 3, \tag{19}$$

i.e. when we eliminate the following cases: $(k, m) \in \{(2, 4), (2, 5), (3, 5), (3, 6)\}$, then $\mathfrak{M}$ satisfies conditions (A1)–(A4) and (12).

Note that in $\mathfrak{M}$ stars are not uniquely determined by star flats. Contrariwise to projective Grassmann spaces a star and a top can meet in a point or in a line. Two distinct tops can share at most a point whereas two distinct stars can share a point, a line, a plane and so on up to a common hyperplane.

Let us start by collecting some technical observations.

Fact 3.1. *Two distinct lines $L_i = \mathbf{p}(H_i, B_i)$, $i = 1, 2$ are coplanar in $\mathfrak{M}$ iff they share some point U and*

(i) *either $H_1 = H_2$, $B_1 \perp B_2$ and $U = B_1 \cap B_2$,*
(ii) *or $B_1 = B_2$ and $U = H_1 + H_2$.*

Fact 3.2. *If Y is maximal totally isotropic and $U \perp Y$, then $U \subseteq Y$.*

Lemma 3.3. *Assume that $X_i = [\Theta, B_i]_k$ for $i = 1, 2$ are two distinct tops. Then $\Upsilon(K_1, K_2)$ iff $U_1 = U_2$ and $B_1 \perp B_2$.*

Proof. $\Rightarrow$: As $X_1 \cap X_2$ is at most a point, by Lemma 1.5(v) and (iii) we have $U_1 = U_2$. A line in K_1 is coplanar with some line in K_2, hence by Fact 3.1 we get $B_1 \perp B_2$.

$\Leftarrow$: Set $U := U_1 = U_2$. According to Fact 3.1, for a line $L_1 = \mathbf{p}(H, B_1)$ in K_1, where $H \in \mathrm{Sub}_{k-1}(U)$, the line $L_2 = \mathbf{p}(H, B_2)$ is a line in K_2 coplanar with L_1. So, we have $\Upsilon(K_1, K_2)$. □

Lemma 3.4. *Assume that $X_i = [H_i, Y_i]_k$ for $i = 1, 2$ are two distinct stars. Then $\Upsilon(K_1, K_2)$ iff either,*

(i) *$H_1 \neq H_2$, $Y_1 = Y_2$, and $U_1 = U_2$, or*
(ii) *$H_1 = H_2$ and one of the following holds:*

(a) *$U_1 \notin X_2$, U_1, U_2 are collinear in $\mathfrak{M}$, and $X_1 \cap X_2$ is a hyperplane in X_1,*
(b) *$X_1 \cap X_2$ contains the line $\overline{U_1, U_2}$ of $\mathfrak{M}$,*
(c) *$X_1 \cap X_2$ contains the point $U_1 = U_2$.*

Proof. $\Rightarrow$: When $H_1 \neq H_2$ the intersection $X_1 \cap X_2$ is at most a point, so by Lemma 1.5(ii) it is a point. Therefore, in view of Lemma 1.5(v) we have $U_1 \in X_2$ and hence, by Lemma 1.5(iii) we get $U_1 = U_2 =: U$.

Now, take any $B_1 \in [U, Y_1]_{k+1}$. The line $L_1 = \mathbf{p}(H_1, B_1) \in K_1$ is coplanar with some line $L_2 = \mathbf{p}(H_2, B_2) \in K_2$, where $B_2 \in [U, Y_2]_{k+1}$. Since $H_1 \neq H_2$, by Fact 3.1 we get $B_1 = B_2$. As the elements of $[U, Y_1]_{k+1}$ span Y_1 we get $Y_1 = Y_2$.

If $H_1 = H_2 =: H$, then X_1, X_2 are contained in the star $[H, V]_k$ of $\mathbf{P}_k(V)$. When $U_1 \neq U_2$, then by Lemma 1.5(iv), (v) we get (a) and by Lemma 1.5(iii) we get (b). Otherwise we get (c).

$\Leftarrow$: In the case of (i) for a line $L_1 = \mathbf{p}(H_1, B)$ in K_1 we take the line $L_2 = \mathbf{p}(H_2, B)$ in K_2. By Fact 3.1 these are coplanar in $\mathfrak{M}$ and consequently $\Upsilon(K_1, K_2)$.

In the case of (ii)(a), as $X_1 \cap X_2$ is a hyperplane it meets any line $L_1 \in K_1$ in some point W, and consequently U_1, W, U_2 span a plane in $\mathfrak{M}$. Thus $\Upsilon(K_1, K_2)$.

For (ii)(b) it is enough to see that every line in K_1 is coplanar with the line $\overline{U_1, U_2}$, which suffices to state that $\Upsilon(K_1, K_2)$.

In (ii)(c) set $H := H_1 = H_2$, $U := U_1 = U_2$, and consider a point $W_1 \in X_1 \backslash X_2$. Note that $\mathfrak{M}$ restricted to $[H, V]_k$ is a polar space P and X_1, X_2 are subspaces of P. Hence $\mathrm{Sub}_k(W_1^{\perp})$ is the hyperplane of points collinear with W_1 in P. So, $\mathrm{Sub}_k(W_1^{\perp}) \cap X_2$ is a hyperplane in X_2. Take $W_2 \in \mathrm{Sub}_k(W_1^{\perp}) \cap X_2$ with $W_2 \neq U$. The points W_1, U, W_2 span a plane in $\mathfrak{M}$ so, the line $\overline{U, W_2}$ is coplanar with $\overline{U, W_1}$.

For $W_1 \in X_1 \cap X_2$ with $W_1 \neq U$ we have $\overline{U, W_1} \in K_1 \cap K_2$. In result $\Upsilon(K_1, K_2)$. □

We follow the same idea as in the case of projective Grassmann spaces and group lines into bundles whose vertices can be uniquely determined. Thus, we have to eliminate cases (ii)(a) and (ii)(b) in Lemma 3.4. Note that (ii)(a) together with Lemma 1.5(iii) shows that Υ need not be symmetric in polar Grassmann spaces. The condition in (16) that makes $\Upsilon_\emptyset$ symmetric rules out (ii)(a) as $U_1, U_2 \in X_1 \cap X_2$ by Lemma 1.5(iii). The requirement that $K_1 \cap K_2 = \emptyset$ rules out (ii)(b).

Lemma 3.5. *Assume that $X_i = [H_i, Y_i]_k$ is a star for $i = 1, 2$ and $X_1 \neq X_2$. Then $\Upsilon_\emptyset(K_1, K_2)$ iff $U_1 = U_2$ is the only point in $X_1 \cap X_2$ and if $H_1 \neq H_2$, then $Y_1 = Y_2$.*

Proof. Follows by Lemma 3.4. □

Let $\overline{\Upsilon_\emptyset}$ be the transitive closure of $\Upsilon_\emptyset$.

Lemma 3.6. *Assume that X_1, X_2 are two distinct tops. Then $\overline{\Upsilon_\emptyset}(K_1, K_2)$ iff $U_1 = U_2$.*

Proof. $\Rightarrow$: In view of Lemma 1.6 and the requirements of (16) all the intermediate cliques are top-cliques. Hence Lemma 3.3 gives $U_1 = U_2$.

$\Leftarrow$: Set $U := U_1 = U_2$ and assume that $X_i = [\Theta, B_i]_k$ for some $B_i \in \mathrm{Q}_{k+1}$. If $B_1 \perp B_2$ we are done by Lemma 3.3. In the case where $B_1 \not\perp B_2$ consider a polar space induced by ξ on $[U, V]_{k+1}$. It is connected, so there is a sequence $W_0, W_1, \ldots W_r$ in $[U, V]_{k+1}$ such that $B_1 = W_0$, $B_2 = W_r$ and $W_{i-1} \perp W_i$ for $i = 1, \ldots, r$. By Lemma 3.3 we have $\Upsilon(K_{i-1}, K_i)$ for $K_i = \mathrm{S}_{[\Theta, W_i]_k}(U)$, $i = 1, \ldots, r$, which completes the proof. □

Now, we prove an analogue of Lemma 2.2 for polar Grassmann spaces.

Lemma 3.7. *If U is a point and X is a maximal strong subspace with $U \in X$, then*

$$\Lambda_{\overline{\Upsilon_\emptyset}}(\mathrm{L}_U(X))) = \{L \in \mathcal{L} \colon U \in L\}. \tag{20}$$

Proof. Set $K := \mathrm{L}_U(X)$. We need to show that $\Lambda_{\overline{\Upsilon_\emptyset}}(K)$ contains all the lines through U. So, let L be a line through U. Consider the maximal strong subspace X' of the same type as X that contains L. Set $K' := \mathrm{S}_{X'}(U)$. It is clear that $L \in K'$. If $X' = X$ then $K' = K$ and thus $L \in \Lambda_{\overline{\Upsilon_\emptyset}}(K)$. So, assume that $X' \neq X$.

In the case where X, X' are tops, we simply apply Lemma 3.6 to get $\overline{\Upsilon_\emptyset}(K, K')$.

So, assume that X, X' are stars. Let $X = [H, Y]_k$ and $L = \mathbf{p}(H', B')$ for some $H, H' \in \mathrm{Q}_{k-1}$, $B' \in \mathrm{Q}_{k+1}$, and $Y \in \mathrm{Q}_m$. Then $X' = [H', Y']_k$ for some $Y' \in \mathrm{Q}_m$. If $H = H'$, then by Lemma 3.5 we have $\Upsilon_\emptyset(K, K')$. If $H \neq H'$ assume additionally that $Y \neq Y'$ as otherwise we get $\Upsilon_\emptyset(K, K')$ by Lemma 3.5. Note that $H, H' \subset U \subset Y, Y'$. Consider the star $X_0 = [H, Y']_k$ and the semibundle $K_0 = \mathrm{S}_{X_0}(U)$. Again by Lemma 3.5 we have $\Upsilon_\emptyset(K, K_0)$ and $\Upsilon_\emptyset(K_0, K')$. Hence, $\overline{\Upsilon_\emptyset}(K, K')$.

In either case $K' \subset \Lambda_{\overline{\Upsilon_\emptyset}}(K)$ by (17) and we are through. □

Like in projective Grassmann spaces here $\Lambda_{\overline{\Upsilon_\emptyset}}(K)$ is also a bundle for all $K \in \mathcal{B}$. In this way the other part of our main Theorem 1.1 for polar Grassmann spaces has been proved.

4. Special cases when semibundles fail

We are going to show that in cases excluded by (15) and (19) where stars and tops are of different dimension Theorem 1.1 remains true.

4.1. Projective Grassmann spaces

Let $\mathfrak{M}$ be a projective Grassmann space $\mathbf{P}_2(V)$ where $n = 5$. In $\mathfrak{M}$ tops are planes. This is dual to $\mathbf{P}_3(V)$ where stars are planes. The intersection of two distinct $\boldsymbol{\pi}$-cliques is an empty set, a line or a pencil of lines. Therefore, in this case we have

$$\mathcal{P} = \{K_1 \cap K_2 \colon K_1, K_2 \in \mathcal{K},\ K_1 \neq K_2,\ |K_1 \cap K_2| \geq 2\}. \tag{21}$$

Top semibundles are not maximal $\boldsymbol{\pi}$-cliques here. According to (13) the family of all top flats can be characterized as

$$\mathcal{K}_{\mathcal{F}} := \{K \in \mathcal{K} : \text{there is no } q \in \mathcal{P} \text{ such that } q \subset K\} = \mathcal{K} \backslash \mathcal{K}_{\mathcal{P}}. \tag{22}$$

Being planes, tops can be identified with top flats in $\mathcal{K}_{\mathcal{F}}$. Actually, top flats, tops, and 3-subspaces of V can all be identified. For flats $F_1, F_2 \in \mathcal{K}_{\mathcal{F}}$ consider the adjacency relation

$$F_1 \sim F_2 \quad \text{iff} \quad \text{there are } q \in \mathcal{P} \text{ and } L_1, L_2 \in q \text{ such that } L_1 \in F_1, L_2 \in F_2. \tag{23}$$

The standard adjacency of points U_1, U_2 in a projective Grassmann space $\mathbf{P}_k(V)$ is the relation

$$U_1 \sim U_2 \quad \text{iff} \quad \dim(U_1 \cap U_2) = k - 1. \tag{24}$$

Lemma 4.1. *Let $F_i = L([\Theta, B_i]_2) \in \mathcal{K}_{\mathcal{F}}$ for $B_i \in \mathrm{Sub}_3(V)$, $i = 1, 2$. Then $F_1 \sim F_2$ iff $B_1 \sim B_2$ as points in $\mathbf{P}_3(V)$.*

Proof. $\Rightarrow$: The pencil of lines q which we have by (23) is extensible to some star semibundle and thus to a star S. Moreover, q determines a plane E contained in S. Say, $E = [H, W]_2$ for some $H \in \mathrm{Sub}_1(V)$ and $W \in \mathrm{Sub}_4(V)$ with $H \subset W$. The lines $L_1, L_2 \in q$ provided by (23) are of the form $L_i = \mathbf{p}(H, B_i)$, $i = 1, 2$. Since $B_1, B_2 \subset W$, considering the dimensions, we have $B_1 \sim B_2$.

$\Leftarrow$: Take $W := B_1 + B_2 \in \mathrm{Sub}_4(V)$, $U := B_1 \cap B_2$, and $H \in \mathrm{Sub}_1(U)$. That way we get the plane $E = [H, W]_2$ which is contained in some star and which shares the line $L_i = \mathbf{p}(H, B_i)$ with F_i for $i = 1, 2$. These lines lie in the required pencil of lines through the point U on the plane E. □

In consequence of Lemma 4.1 the structure $\langle \mathcal{K}_{\mathcal{F}}, \sim \rangle$ is the structure of adjacency in $\mathbf{P}_3(V)$. By Chow's Theorem [6] we can reconstruct the underlying projective space $\mathbf{P}_1(V)$ in $\mathbf{P}_3(V)$. This proves that $\mathfrak{M}$ is definable in $\langle \mathcal{L}, \boldsymbol{\pi} \rangle$ and hence Theorem 1.1 remains true in this specific case.

The same reasoning applies to the dual case of $\mathbf{P}_3(V)$.

4.2. Polar Grassmann spaces

The reasoning for top flats in a projective Grassmann space from Sect. 4.1 can be applied also to a polar Grassmann space $\mathfrak{M} := \mathbf{P}_2(\mathrm{Q})$ where $m = 5$

($n \geq 10$) with only a small change to the definition of adjacency of points in $\mathfrak{M}$. Adjacency of points U_1, U_2 in a polar Grassmann space $\mathbf{P}_k(\mathrm{Q})$ differs from that in (24) and is given by

$$U_1 \sim U_2 \quad \text{iff} \quad \dim(U_1 \cap U_2) = k - 1 \text{ and } U_1 \perp U_2. \tag{25}$$

This causes that in Lemma 4.1 and its proof all the subspaces of V, including the subspace W that appears in the plane $E = [H, W]_2$, must be totally isotropic. Then Lemma 4.1 reads as follows.

Lemma 4.2. *Let $F_i = L([\Theta, B_i]_2) \in \mathcal{K}_{\mathcal{F}}$ for $B_i \in \mathrm{Q}_3$, $i = 1, 2$. Then $F_1 \sim F_2$ iff $B_1 \sim B_2$ as points in $\mathbf{P}_3(\mathrm{Q})$.*

So, $\langle \mathcal{K}_{\mathcal{F}}, \sim \rangle$ is the structure of adjacency in $\mathbf{P}_3(\mathrm{Q})$ where, by Chow's Theorem for polar Grassmann spaces proved in [26], we can reconstruct the underlying polar space $\mathbf{P}_1(\mathrm{Q})$. This proves Theorem 1.1 in this case.

Now, consider $\mathfrak{M} := \mathbf{P}_3(\mathrm{Q})$ for $m = 5$, where stars are planes. In this case equation (21) is also true whereas $\mathcal{K}_{\mathcal{F}}$ in (23) is the set of all star flats. Generally, the idea is the same as in the other cases when semibundles fail: bring the adjacency on star flats to a well known adjacency in some polar Grassmann space and apply a suitable version of Chow's Theorem. The problem here is that we cannot simply identify a star $[H, Y]_3$ with its vertex H as we did earlier. First we need to identify all the stars with the same vertex.

A star flat as the set of all lines in a star can be identified with that star. For $F_1, F_2 \in \mathcal{K}_{\mathcal{F}}$ we define

$$F_1 \ \sigma \ F_2 \quad \text{iff} \quad F_1 = F_2 \text{ or } F_1 \cap F_2 \in \mathcal{L}. \tag{26}$$

This relation is not transitive. Let δ be the transitive closure of σ. Hence δ is an equivalence relation. Set $\mathrm{S}(H) := \big\{\mathrm{L}\big([H, Y]_3\big) : Y \in \mathrm{Q}_5\big\}$ for $H \in \mathrm{Q}_2$.

Lemma 4.3. *If $K = \mathrm{L}\big([H, Y]_3\big) \in \mathcal{K}_{\mathcal{F}}$, then $[K]_\delta = \mathrm{S}(H)$.*

Proof. It suffices to show that given two stars $S_i := [H, Y_i]_3$, for $i = 1, 2$ we will have $\mathrm{L}(S_1) \ \delta \ \mathrm{L}(S_2)$. So, consider a sequence $D_0, D_1, \dots, D_r \in \mathrm{Q}_5$ such that $D_1 = Y_1$, $D_r = Y_2$, and $\dim(D_{i-1} \cap D_i) = 4$ for $i = 1, \dots, r$. Note that the star $[H, D_{i-1}]_3$ shares the line $\mathbf{p}(H, D_{i-1} \cap D_i)$ with the star $[H, D_i]_3$ which completes the proof by (26). $\square$

In other words δ identifies stars with the same vertex. For $K_1, K_2 \in \mathcal{K}_{\mathcal{F}}$ we define adjacency on equivalence classes of δ as follows

$$\begin{aligned} [K_1]_\delta \approx [K_2]_\delta \quad \text{iff} \quad & \text{there are } F_1, F_2 \in \mathcal{K}_{\mathcal{F}} \quad \text{with} \\ & F_1 \sim F_2,\ F_1 \ \delta \ K_1, \text{ and } F_2 \ \delta \ K_2. \end{aligned} \tag{27}$$

Lemma 4.4. *Let $H_1, H_2 \in \mathrm{Q}_2$. Then $\mathrm{S}(H_1) \approx \mathrm{S}(H_2)$ iff $H_1 \sim H_2$.*

Proof. $\Rightarrow$: According to (27), there are stars $S_i = [H_i, Y_i]_3$ such that $L(S_i) \in \mathrm{S}(H_i)$ for $i = 1, 2$, and $L(S_1) \sim L(S_2)$. Then, by (23), S_1 meets S_2 in a point, which implies $H_1 \sim H_2$.

$\Leftarrow$: We take $U := H_1 + H_2$ and $B \in \mathrm{Q}_4$, $Y \in \mathrm{Q}_5$ such that $U \subset B \subset Y$. Let us consider the plane $E = [H_1 \cap H_2, B]_3$, which is contained in the top $[\Theta, B]_3$. Then E shares the line $L_i = \mathbf{p}(H_i, B)$ with the star $S_i = [H_i, Y]_3$ for $i = 1, 2$. Both L_1, L_2 are in the pencil of lines through the point U on the plane E. So, we have $L(S_1) \sim L(S_2)$, and consequently $\mathrm{S}(H_1) \approx \mathrm{S}(H_2)$ by (27). $\square$

Summing up, again by Chow's Theorem for polar Grassmann spaces the underlying polar space can be reconstructed in the structure of adjacency in $\mathbf{P}_2(\mathrm{Q})$ which is $\langle \mathcal{K}_{\mathcal{F}}/\delta, \approx \rangle$. This means that Theorem 1.1 holds true in this case as well.

4.3. Final remarks

Loosely speaking, in this paper semibundles are grouped into bundles of lines, which are then used to reconstruct points of a projective or a polar Grassmann space. In a natural way connections with the Bundle Theorem appear.

Theorem 4.5. (The Bundle Theorem) *If L_1, L_2, L_3, L_4 are lines such that no three of them are coplanar, and five of the six pairs $\{L_i, L_j\}$, $1 \le i < j \le 4$ are coplanar, then so is the sixth pair.*

Most of the time, e.g. in [1,15,18,34], the Bundle Theorem is the necessary condition for the construction of the bundle space in locally projective spaces. We do not follow exactly this path. Nevertheless, projective and polar Grassmann spaces are locally projective in the sense that their maximal strong subspaces are projective spaces. Moreover, projective Grassmann spaces satisfy Theorem 4.5, which is proved below.

Proof of Theorem 4.5 for projective Grassmann spaces. Assume that all the pairs, except L_3, L_4, of the four lines are coplanar. Consider the lines L_1, L_2. As they are coplanar they share a point, say U. The line L_3 is coplanar with both L_1 and L_2. Hence it meets L_1 and L_2. So the lines L_1, L_2, L_3 form a triangle or they all go through the point U. In the first case the assumption that no three lines of the four are coplanar is violated. Therefore, $U \in L_1, L_2, L_3$. By the same argument $U \in L_4$. Note that the lines L_1, L_2, L_3 and L_1, L_2, L_4 form $\boldsymbol{\pi}$-cliques, and by Lemma 1.3 as semibundles lie in maximal strong subspaces, say X and Y respectively. Then X and Y share the plane spanned by L_1, L_2, and thus $X = Y$. Finally, L_3, L_4 are contained in one maximal strong subspace, and therefore they are coplanar. $\square$

The situation is different in the case of polar Grassmann spaces. Let $\mathfrak{M} := \mathbf{P}_k(\mathrm{Q})$ be a polar Grassmann space with $m \ge k + 3$. Take two distinct stars

$S_i = [H_i, Y_i]_k$, $i = 1, 2$, which share a plane E. Let U be a point in E and L_1, L_2 be two distinct lines through the point U on the plane E. Take two points $U_1 \in S_1 \backslash S_2$, $U_2 \in S_2 \backslash S_1$ and consider lines $L_3 = \overline{U_1, U}$, $L_4 = \overline{U_2, U}$. Note that $L_3 = \mathbf{p}(H_1, U + U_1)$ and $L_4 = \mathbf{p}(H_2, U + U_2)$. So, in view of Fact 3.1 L_3 is not coplanar with L_4.

Remark 4.6. *In general, the Bundle Theorem is false in polar Grassmann spaces due to the fact that two distinct stars can share a plane.*

Acknowledgements

We would like to thank Krzysztof Prażmowski for his support and fruitful discussions, that considerably improved the paper.

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Received: June 8, 2015
Revised: September 28, 2015